

THE GENERATING RANK OF THE UNITARY AND SYMPLECTIC GRASSMANNIANS

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ABSTRACT. We prove that the Grassmannian of totally isotropic k -spaces of the polar space associated to the unitary group $\mathrm{SU}_{2n}(\mathbb{F})$ ($n \in \mathbb{N}$) has generating rank $\binom{2n}{k}$ when $\mathbb{F} \neq \mathbb{F}_4$. We also reprove the main result of Blok [4], namely that the Grassmannian of totally isotropic k -spaces associated to the symplectic group $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $\binom{2n}{k} - \binom{2n}{k-2}$, when $\mathrm{Char}(\mathbb{F}) \neq 2$.

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1. INTRODUCTION

Generating sets of point-line geometries serve both theoretical and computational purposes. For instance, if the geometry admits a finite-dimensional absolutely universal embedding and if the generating rank equals the (vector) dimension of a particular embedding, then the latter is absolutely universal. On the other hand, minimal generating sets may serve in creating computer models of point-line geometries.

Until now, the projective building of type A_n associated to the group $\mathrm{SL}_{n+1}(\mathbb{F})$ and the symplectic building of type C_n associated to $\mathrm{Sp}_{2n}(\mathbb{F})$ ($\mathrm{Char}(\mathbb{F}) \neq 2$) are the only ones of which the generating rank is known for each of its k -shadow spaces (k a single node of the diagram). (see Cooperstein and Shult [14], Blok and Brouwer [6], Blok [3, 4], Cooperstein [12, 13], and De Bruyn and Pasini [18].)

In this paper we deal with the unitary building of type ${}^2A_{2n-1}(\mathbb{F})$ associated to the group $\mathrm{SU}_{2n}(\mathbb{F})$ over a field $\mathbb{F} \not\cong \mathbb{F}_4$ and exhibit minimal generating sets for all associated unitary k -Grassmannians. For a few of these geometries a minimal generating set has already been found. Since the polar space associated to $\mathrm{SU}_4(q^2)$ has more points per line than lines per point, it is generated by the four points of an apartment. A result by Blok and Brouwer [6, Theorem 2.1] then implies that the generating rank of the unitary polar space associated to the group $\mathrm{SU}_{2n}(\mathbb{F}_{q^2})$ is $2n$.

In [11] Cooperstein proved that the dual polar space associated to $SU_{2n}(q^2)$ has generating rank $\binom{2n}{n}$ when $q^2 > 2$. This result was generalized to include infinite fields by De Bruyn and Pasini [18]. It was proved by A. E. Brouwer that the generating rank is at least $\frac{4n+2}{3}$ when $q^2 = 4$. His conjecture that equality holds was confirmed by P. Li [22].

We shall employ these results to obtain minimal generating sets for all k -shadow spaces of this building in a unified way.

Theorem 1.1. *The k -Grassmannian of the polar space associated to $SU_{2n}(\mathbb{F})$ (\mathbb{F} a field) has generating rank $\binom{2n}{k}$ if $\mathbb{F} \neq \mathbb{F}_4$ or $k = 1$.*

The techniques we use also allow us to describe generating sets for all symplectic k -Grassmannians over fields of characteristic different from 2 in almost exactly the same manner, thus giving an alternative proof for the main result in Blok [4].

Theorem 1.2. (Blok [4]) *The k -Grassmannian of the polar space associated to $Sp_{2n}(\mathbb{F})$ (\mathbb{F} a field) has generating rank $\binom{2n}{k} - \binom{2n}{k-2}$ if $\text{Char}(\mathbb{F}) \neq 2$.*

More on generating sets can be found in Cooperstein [13] and Blok [2].

In Section 2 we define some basic concepts, including the generating rank of a point-line geometry. We also introduce all geometries under study. We then prove Proposition 2.1, which is a special case of Theorems 1.1 and 1.2, and present Theorem 2.3, which is a special case of Theorem 1.1. These two will form the basis of an inductive proof of our main results. In Section 3 we describe embeddings of Γ_k^f of dimension d_k , where

$$d_k = \begin{cases} \binom{2n}{k} & \text{if } f=h \\ \binom{2n}{k} - \binom{2n}{k-2} & \text{if } f=s \end{cases}$$

This section also contains a short proof of the probably well-known result that the $SU(V)$ module $\bigwedge^k V$ is irreducible when $k < \dim(V)$ (see Theorem 3.1).

It then suffices (see (1)) to show that Γ_k^f has a generating set of size exactly d_k . We do this in Section 5. The definition of the generating set, as well as the proof that the proposed set generates the geometry and has the desired cardinality is inductive. Two inductive tools are described in Section 4.

2. PRELIMINARIES

A *point-line geometry* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ where \mathcal{P} is a set whose elements are called ‘points’ and \mathcal{L} is a collection of subsets of \mathcal{P} called ‘lines’ with the property that any two distinct points belong to at most

one line. If \mathcal{P} and \mathcal{L} are not mentioned explicitly, the sets of points and lines of a point-line geometry Γ are denoted $\mathcal{P}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

A *subspace* of Γ is a subset $X \subseteq \mathcal{P}$ such that any line containing at least two points of X is itself entirely contained in X . We call X *proper* if $X \subsetneq \mathcal{P}$. A *hyperplane* of Γ is a proper subspace that meets every line.

Projective embeddings and generating sets. The *span* of a set $\mathcal{S} \subseteq \mathcal{P}$ is the smallest subspace of Γ containing \mathcal{S} ; it is the intersection of all subspaces containing \mathcal{S} and is denoted by $\langle \mathcal{S} \rangle_\Gamma$. We say that \mathcal{S} is a *generating set* (or *spanning set*) for Γ if $\langle \mathcal{S} \rangle_\Gamma = \mathcal{P}$.

For a vector space W over some field \mathbb{F} , the *projective geometry* associated to W is the point-line geometry $\mathbb{P}(W) = (\mathcal{P}(W), \mathcal{L}(W))$ whose points are the 1-spaces of W and whose lines are the sets of 1-spaces contained in some 2-space.

A *projective embedding* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a pair (ϵ, W) , where ϵ is an injective map $\mathcal{P} \xrightarrow{\epsilon} \mathcal{P}(W)$ that sends every line of \mathcal{L} onto a line of $\mathcal{L}(W)$, and with the property that

$$\langle \epsilon(\mathcal{P}) \rangle_{\mathbb{P}(W)} = \mathcal{P}(W).$$

In the literature, this is sometimes referred to as a *strong* or *full* projective embedding. The *dimension* of the embedding is the dimension of the vector space W . In this paper we will assume both $\dim(W)$ and $|\mathcal{S}|$, (\mathcal{S} as above) to be finite. Then, since $\epsilon(\langle \mathcal{S} \rangle_\Gamma) \subseteq \langle \epsilon(\mathcal{S}) \rangle_W$, for any generating set \mathcal{S} and any embedding (ϵ, W) we have

$$(1) \quad \dim(W) \leq |\mathcal{S}|.$$

In case of equality, \mathcal{S} is a minimal generating set and (ϵ, W) is an embedding of maximal dimension. We then call $\dim(W) = |\mathcal{S}|$ the *generating rank* of Γ .

We briefly describe the particular geometries we will discuss in this paper.

The Projective Grassmannian. Let V be a vector space over some field \mathbb{F} . For any k with $1 \leq k \leq \dim(V) - 1$, the *projective k -Grassmannian* associated to V is the point-line geometry $\text{Gr}(V, k)$ whose points are the k -spaces of V and whose lines are sets of the form

$$\{K \text{ a } k\text{-space in } V \mid D \subseteq K \subseteq U\}$$

for some $(k-1)$ -space D and $(k+1)$ -space $U \supseteq D$. One verifies that the map

$$(2) \quad \begin{array}{ccc} \mathbf{e}_k: \text{Gr}(V, k) & \rightarrow & \text{PG}(\bigwedge^k V) \\ U & \mapsto & \bigwedge^k U \end{array}$$

defines a full projective embedding for $\text{Gr}(V, k)$, called the *Grassmann embedding*. Grassmann embeddings have been studied in connection with many topics, including hyperplanes [23], highest weight modules [5], algebraic varieties [10] and Schubert Calculus [20].

The Unitary and Symplectic Grassmannians. Let V be a vector space of dimension $2n$ over the field \mathbb{F} endowed with a form \mathbf{f} . We assume that either \mathbf{f} is non-degenerate and symplectic, or \mathbf{f} is non-degenerate σ -Hermitian of Witt index n . This means that σ is the generator of the Galois group of a quadratic field extension \mathbb{F}/\mathbb{K} , which is assumed to be separable in the case the characteristic is two. The norm $N_\sigma: \mathbb{F} \rightarrow \mathbb{K}$ is then trace-valued. We sometimes write $\mathbb{K} = \mathbb{F}^\sigma$ to emphasize that \mathbb{K} is the fixed subfield of \mathbb{F} under σ . In the respective cases we sometimes write \mathbf{s} or \mathbf{h} for \mathbf{f} . For a subspace U of V we define

$$U^\perp = \{v \in V \mid \mathbf{f}(u, v) = 0 \forall u \in U\}.$$

We write $U \perp W$ if $W \subseteq U^\perp$. The *radical* of a subspace W is

$$\text{Rad}(W, \mathbf{f}) = W^\perp \cap W.$$

A subspace U of V is called *totally isotropic* (t.i.) with respect to the form $\mathbf{f}(\cdot, \cdot)$ if $U \subseteq U^\perp$. It is called non-degenerate if $\text{Rad}(U, \mathbf{f}) = \{0\}$.

The *polar building* Γ associated to \mathbf{f} is the spherical building whose objects of type $i \in I = \{1, 2, \dots, n\}$ are the i -spaces in V that are totally isotropic with respect to \mathbf{f} . Two objects are incident when one contains the other as a subspace of V (see e.g. [9, 26, 27]).

The *polar space* associated to \mathbf{f} is the point-line geometry $\Gamma_1^\mathbf{f}$ whose points are the t.i. 1-spaces of V and whose lines are sets of 1-spaces of the form

$$\{P \text{ a t.i. 1-space of } V \mid P \subseteq L\}$$

for some t.i. 2-space L . We sometimes call t.i. 3-spaces *planes*. A *hyperbolic line* H is a 2-space in V such that \mathbf{f} restricted to H is non-degenerate of Witt index 1. Since both the symplectic and unitary polar space under study in this paper have Witt index $n = \frac{1}{2} \dim(V)$, the vector space V has a basis $\mathcal{E} = \{e_i, e_{n+i} \mid i = 1, 2, \dots, n\}$ that is *hyperbolic* with respect to \mathbf{f} . That is, $\mathbf{f}(e_i, e_j) = 0$ for all $1 \leq i \leq j \leq 2n$ with $j \neq n+i$ and $\mathbf{f}(e_i, e_{n+i}) = 1$ for all $i = 1, 2, \dots, n$. Each hyperbolic basis \mathcal{E} for V gives rise to an apartment $\Sigma(\mathcal{E})$ of the polar building; it can conveniently be described as the collection $\Sigma(\mathcal{E}) = \{E_{J,K} \mid J, K\}$, where $E_{J,K} = \langle e_j, e_{k+n} \mid j \in J, k \in K \rangle$ and (J, K) runs over all pairs of subsets of $\{1, 2, \dots, n\}$ such that $J \cap K = \emptyset \neq J \cup K$.

The (*polar*) k -*Grassmannian* associated to \mathbf{f} is the point-line geometry $\Gamma_k^\mathbf{f}(V) = (\mathcal{P}_k, \mathcal{L}_k)$ whose points are the t.i. k -spaces and whose lines

are the sets of the form

$$\{K \text{ a t.i. } k\text{-space in } V \mid D \subseteq K \subseteq U\}$$

for some t.i. $(k-1)$ -space D and t.i. $(k+1)$ -space $U \supseteq D$. In case $k = n$, the lines are of the form

$$\{K \text{ a t.i. } k\text{-space in } V \mid D \subseteq K\}$$

for some t.i. $(n-1)$ -space D . Whenever V or \mathbf{f} is clear from the context, we'll drop it from the notation. Note that $\Gamma_k^{\mathbf{f}}(V)$ is a subgeometry of $\text{Gr}(V, k)$, i.e. $\mathcal{P}(\Gamma_k^{\mathbf{f}}) \subseteq \mathcal{P}(\text{Gr}(V, k))$ and $\mathcal{L}(\Gamma_k^{\mathbf{f}}) \subseteq \mathcal{L}(\text{Gr}(V, k))$.

The k -residue or simply *residue in Γ_k* of an object X of Γ , is denoted $\text{Res}_k(X)$ and is the subgeometry of points and lines of Γ_k that are incident to X when viewed in Γ .

We denote the group of linear transformations of determinant 1 preserving the form \mathbf{f} on V (i.e. the symmetry group of \mathbf{f}) by $\text{SG}(V, \mathbf{f})$. Thus, $\text{SG}(V, \mathbf{s}) = \text{Sp}(V) \cong \text{Sp}_{2n}(\mathbb{F})$ and $\text{SG}(V, \mathbf{h}) = \text{SU}(V) \cong \text{SU}_{2n}(\mathbb{F})$. Clearly $\text{SG}(V, \mathbf{f})$ is an automorphism group of $\Gamma_k^{\mathbf{f}}$ for all $k = 1, 2, \dots, n$ in that it preserves points and lines and the incidence between them.

The following special case of Theorems 1.1 and 1.2 will be instrumental in proving those theorems.

Proposition 2.1. *Let Γ_1 be the polar space associated to a non-degenerate symplectic or Hermitian form \mathbf{f} of Witt index n on a vector space V of dimension $2n$ over \mathbb{F} . Moreover, in case the form is symplectic assume that $\text{Char}(\mathbb{F}) \neq 2$. Then Γ_1 has generating rank $2n$ and it is generated by the $2n$ points of an apartment i.e. a hyperbolic basis.*

Proof. Theorem 2.1 of [6] shows that a non-degenerate polar space of rank $n \geq 2$ is generated by the $2n$ points of an apartment if this is so in rank 2. For the symplectic rank-2 case this is proved in e.g. loc. cit. or [14]. For the Hermitian rank-2 case this is the content of Lemma 2.2 below. \square

Lemma 2.2. *The unitary rank 2 polar space associated to $\text{SU}_4(\mathbb{F})$ is spanned by four points.*

Proof. Let \mathbb{F} , \mathbb{K} and σ be as defined above. For convenience denote images under σ using the bar notation. Assume \mathbb{F} is generated over \mathbb{K} by the element δ which we further assume satisfies $\delta + \bar{\delta} = 0$ if the characteristic is not two.

We assume that $\Gamma_1 = (\mathcal{P}, \mathcal{L})$ is the polar space associated to the σ Hermitian form $\mathbf{h}: V \times V \rightarrow \mathbb{F}$, where V is a 4-dimensional space over \mathbb{F} . Let $\mathcal{E} = \{e_i, e_{2+i} \mid i = 1, 2\}$ be a hyperbolic basis for V with respect to \mathbf{h} . We will prove that the set $\mathcal{S} = \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_4 \rangle\}$

is a generating set for $(\mathcal{P}, \mathcal{L})$. Before proving this we introduce some notation.

Let $L = \langle e_1, e_2 \rangle$, $M = \langle e_3, e_4 \rangle$, $P_a = \langle e_1 + ae_2 \rangle$, and $Q_a = \langle -\bar{a}e_3 + e_4 \rangle$.

The following well-known facts about Γ_1 and the isometry group $\mathrm{SU}_4(\mathbb{F})$ are easy to prove:

- (i) If \mathbf{x}, \mathbf{y} are isotropic vectors and $h(\mathbf{x}, \mathbf{y}) = 1$ then a vector $\mathbf{x} + a\mathbf{y}$ is isotropic if and only if $\mathrm{trace}(a) = a + \bar{a} = 0$. When the characteristic is two this implies that $a \in \mathbb{K}$ and otherwise $a \in \delta\mathbb{K}$.
- (ii) The unique point on M which is orthogonal to P_a is Q_a .
- (iii) If $\tau \in \mathrm{SL}(V)$ and the matrix of τ with respect to \mathcal{B} is

$$\begin{pmatrix} A & 0_{22} \\ 0_{22} & B \end{pmatrix},$$

where A, B are 2×2 matrices, then $\tau \in \mathrm{SU}_4(\mathbb{F})$ if and only if $\bar{B} = A^{-T}$.

- (iv) A consequence of (iii) is that $\mathrm{SU}_4(\mathbb{F})$ is transitive on the points of L .
- (v) A further consequence of (iii) is that $\mathrm{SU}_4(\mathbb{F})$ is transitive on pairs (P, Q) where $P \in L$, $Q \in M$ and P and Q are not orthogonal.

We now prove our assertion about the generation of \mathcal{P} . Let \mathcal{X} denote the subspace of \mathcal{P} generated by \mathcal{S} . Let R be an arbitrary point of \mathcal{P} . If $R \in L \cup M$ there is nothing to prove. Otherwise, there are points $P \in L, Q \in M$ such that $R \in \langle P, Q \rangle$. If $P \perp Q$ then R is in \mathcal{X} since both P and Q are. Thus, we may assume that P and Q are not orthogonal. By (v) we can assume that $P = \langle e_1 \rangle$ and $Q = \langle e_3 \rangle$. By (i) we may therefore assume that $R = \langle e_1 + c\epsilon e_3 \rangle$ for some $c \in \mathbb{K}$ where $\epsilon = 1$ if the characteristic is two and $\epsilon = \delta$ otherwise.

As previously pointed out, the points P_a, Q_a are orthogonal and in \mathcal{X} and therefore the line $\langle P_a, Q_a \rangle$ is contained in \mathcal{X} . In particular, the point $D_a = \langle (e_1 + ae_2) - (c\epsilon/\bar{a})(-\bar{a}e_3 + e_4) \rangle = \langle e_1 + ae_2 + c\epsilon e_3 - (c\epsilon/\bar{a})e_4 \rangle$ is isotropic and in \mathcal{X} . The same holds for $D_b = \langle e_1 + be_2 + c\epsilon e_3 - (c\epsilon/\bar{b})e_4 \rangle$. Using that $\bar{\epsilon} = -\epsilon$ and $\bar{c} = c$, we find

$$\begin{aligned} h(e_1 + ae_2 + c\epsilon e_3 - \frac{c\epsilon}{\bar{a}}e_4, e_1 + be_2 + c\epsilon e_3 - \frac{c\epsilon}{\bar{b}}e_4) = \\ -c\epsilon + \frac{c\epsilon a}{b} + c\epsilon - \frac{c\epsilon \bar{b}}{\bar{a}} = c\epsilon \left(\frac{a}{b} - \frac{\bar{b}}{\bar{a}} \right) \end{aligned}$$

so that $D_a \perp D_b$ if and only if $a\bar{a} = b\bar{b}$.

It is possible to choose distinct a and b such that $a\bar{a} = b\bar{b}$. We claim that $R = \langle e_1 + c\epsilon e_3 \rangle \in \langle D_a, D_b \rangle$. Namely, we have

$$\begin{aligned} b(e_1 + ae_2 + c\epsilon e_3 - \frac{c\epsilon}{a}e_4) - a(e_1 + be_2 + c\epsilon e_3 - \frac{c\epsilon}{b}e_4) = \\ (b-a)e_1 + (b-a)c\epsilon e_3 + (c\epsilon) \left[\frac{a}{b} - \frac{b}{a} \right] e_4 = \\ (b-a)[e_1 + c\epsilon e_3]. \end{aligned}$$

We can conclude from $R \in \langle D_a, D_b \rangle$, $D_a \perp D_b$, and $D_a, D_b \in \mathcal{X}$ that $R \in \mathcal{X}$ as required. \square

We shall also need the following result from De Bruyn and Pasini [18] and Cooperstein [12, 11] on the generating rank of Γ_n .

Theorem 2.3. *Let Γ_n be the dual polar space associated to a non-degenerate symplectic or Hermitian form \mathbf{f} on a vector space V of dimension $2n$ over \mathbb{F} . Then*

- (a) Γ_n has generating rank $\binom{2n}{n}$ if $\mathbf{f} = \mathbf{h}$ is Hermitian and $\mathbb{F} \neq \mathbb{F}_4$, and
- (b) Γ_n has generating rank $\binom{2n}{n} - \binom{2n}{n-2}$ if $\mathbf{f} = \mathbf{s}$ is symplectic and $\mathbb{F} \not\cong \mathbb{F}_2$.

3. EMBEDDINGS

We describe the fairly well-known Grassmann embeddings of the polar Grassmannian $\Gamma_k^{\mathbf{f}}$. We consider arbitrary (\mathbf{f}, k) with $1 \leq k \leq n$ and $\mathbf{f} = \mathbf{s}, \mathbf{h}$, excluding - for the moment - only the case (\mathbf{h}, n) . We then define the following map on the point-set of $\Gamma_k^{\mathbf{f}}$:

$$(3) \quad \begin{aligned} \mathbf{e}_{\text{gr}}: \Gamma_k &\rightarrow \text{PG}(\bigwedge^k V) \\ U &\mapsto \bigwedge^k U \end{aligned}$$

Here for any ordered set of vectors $\{u_1, \dots, u_k\}$ define $\bigwedge^k \{u_1, \dots, u_k\} = u_1 \wedge u_2 \wedge \dots \wedge u_k$. One verifies that, if $\{u_1, \dots, u_k\}$ is a basis for a subspace U , then the 1-space $\bigwedge^k U = \langle \bigwedge^k \{u_1, \dots, u_k\} \rangle$ only depends on U , not on the chosen ordered basis. Thus, \mathbf{e}_{gr} is simply the restriction to Γ_k of the Grassmann embedding defined in (2). Since the lines of Γ_k are lines of $\text{Gr}(V, k)$, it follows that \mathbf{e}_{gr} is in fact an embedding of Γ_k into $\text{PG}(V_{\text{gr}})$ for some subspace V_{gr} of $\bigwedge^k V$.

Theorem 3.1. *Let V be a vector space of dimension n over \mathbb{F} endowed with a non-degenerate Hermitian form. Then, the vector space $W = \bigwedge^k V$ is irreducible as a module for the unitary group $\text{SU}_n(\mathbb{F}) = \text{SU}(V)$.*

Proof. We shall prove that the space $W = \bigwedge^k V$, which has dimension $\binom{2n}{k}$ over \mathbb{F} is an irreducible module for the unitary group $\mathrm{SU}_n(\mathbb{F}) = \mathrm{SU}(V)$. We first note that $\mathrm{SU}(V) \leq \mathrm{SL}(V)$ and that W is irreducible for $\mathrm{SL}(V)$ over any field. This is well-known, but it is also easy to see. View $\mathrm{SL}(V) \cong \mathrm{SL}_n(\mathbb{F})$ via a basis $\{a_1, \dots, a_n\}$ for V and let H be the group of diagonal matrices. Then since H is abelian, any submodule U of W is the direct sum of weight spaces, i.e. common eigenspaces for all elements of H . One verifies that since $|\mathbb{F}| \geq 4$ the decomposition of W into weight spaces for H is unique and equals $\bigoplus_{w \in \mathcal{W}} w$, where $\mathcal{W} = \{\langle a_{i_1} \wedge \dots \wedge a_{i_k} \rangle_W \mid 1 \leq i_1 < \dots < i_k \leq n\}$. Thus, if U is non-zero it contains at least one 1-space from \mathcal{W} , but since the subgroup $N \leq \mathrm{SL}_n(\mathbb{F})$ of monomial matrices is transitive on \mathcal{W} and \mathcal{W} spans W , we find $U = W$.

In view of what we just saw, it suffices to show that if a subspace of W is $\mathrm{SU}(V)$ -invariant, then it is $\mathrm{SL}(V)$ -invariant.

To this end we examine the inclusion $\mathrm{SU}(V) \leq \mathrm{SL}(V)$ a little closer. Each root group T^σ of the former is contained in a unique root group T of the latter. Namely, there is an \mathfrak{h} -isotropic vector u , such that

$$\begin{aligned} T^\sigma &= T_u^\sigma = \langle t_{u,\lambda} \mid \lambda \in \mathbb{F}^\sigma \rangle, \\ T &= T_u = \langle t_{u,\lambda} \mid \lambda \in \mathbb{F} \rangle, \end{aligned}$$

where $t_{u,\lambda}(v) = v + \lambda \delta \mathfrak{h}(v, u)u$ and \mathbb{F} is generated over \mathbb{F}^σ by δ , and moreover, $\delta + \delta^\sigma = 0$ whenever $\mathrm{Char}(\mathbb{F}) \neq 2$. One can show that $\mathrm{SU}(V)$ is generated by all such groups T_u^σ (See e.g. [19, 25]). What is more, we claim that:

$$(4) \quad \mathrm{SL}(V) = \langle T_u \mid u \in V \text{ } \mathfrak{h}\text{-isotropic} \rangle.$$

Proof of (4): For $n = 2$ it is easy to verify that any two non-orthogonal isotropic vectors e and f suffice. For $n \geq 3$, let a_1, \dots, a_n be a basis for V that is orthogonal with respect to \mathfrak{h} . Then

$$(5) \quad \mathrm{SL}(V) = \langle \mathrm{SL}(\langle a_i, a_{i+1} \rangle) \mid 1 \leq i \leq n-1 \rangle.$$

Presenting $\mathrm{SL}(V) \cong \mathrm{SL}_n(\mathbb{F})$ via the basis $\{a_1, \dots, a_n\}$, it is rather easy to show, using commutators of elementary matrices, that the right hand-side of (5) contains all elementary matrices, so that equality follows.

Since the norm $N_\sigma: \mathbb{F} \rightarrow \mathbb{F}^\sigma$ is surjective, for each i there exist two isotropic vectors e_i and f_i such that $\langle a_i, a_{i+1} \rangle = \langle e_i, f_i \rangle$ (see e.g. [25, Ch. 10]). Now (4) follows by applying the $n = 2$ case $n-1$ times.

Next, we compute the action of $(t_{u,\lambda} - \mathrm{id})$ on W . Select a basis $u = v_1, \dots, v_n$ for V such that $\mathfrak{h}(v_i, v_1) = 0$ for all $i \neq 2$ and $\delta \mathfrak{h}(v_2, v_1) = 1$. The set of all pure vectors $\bigwedge_{j=1}^k v_{i_j}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$

forms a basis for W . For such a basis we have

$$\begin{aligned} (t_{u,\lambda} - \text{id})(\wedge_{j=1}^k v_{i_j}) &= \wedge_{j=1}^k t_{v_1,\lambda}(v_{i_j}) - \wedge_{j=1}^k v_{i_j} \\ &= \sum_{j=1}^k v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} \wedge \lambda \delta h(v_{i_j}, v_1) v_1 \wedge v_{i_{j+1}} \wedge \cdots \wedge v_{i_k} \end{aligned}$$

since any pure wedge product with two or more occurrences of v_1 is the zero vector. Using that $h(v_i, v_1) = 0$ for all $i \neq 2$, we find that

$$(t_{u,\lambda} - \text{id})(\wedge_{j=1}^k v_{i_j}) = \begin{cases} \lambda v_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} & \text{if } i_1 = 2, \\ 0 & \text{else.} \end{cases}$$

Since the image of any basis vector is either 0 or again a basis vector multiplied by λ , it follows that for any 1-dimensional subspace A of W we have $[A, T_u] = [A, T_u^\sigma]$. (Recall that if a group T' acts on a vector space V' , then for any subspace $A' \leq V'$ one writes $[A', T'] = \langle (t' - \text{id})(a) \mid t' \in T' \ a \in A' \rangle_{V'}$. See e.g. [21].) Clearly an arbitrary subspace U of W is invariant under T_u^σ if and only if $[A, T_u^\sigma] \leq U$ for any 1-dimensional subspace A of U , and the same holds for T_u . Thus since $[A, T_u^\sigma] = [A, T_u]$, for all such A , U is T_u^σ -invariant if and only if it is T_u -invariant. Our claim now follows from (4). \square

Corollary 3.2. *Let $1 \leq k \leq n-1$. Then, the Grassmann embedding $(e_{\text{gr}}, V_{\text{gr}})$ for Γ_k^h has dimension $\binom{2n}{k}$. More precisely, we have $V_{\text{gr}} = \wedge^k V$.*

Proof. As $\text{SU}(V)$ acts transitively on the t.i. k -spaces in V , the space V_{gr} is an $\text{SU}(V)$ -submodule of $\wedge^k V$. By Theorem 3.1, $V_{\text{gr}} = \wedge^k V$. \square

The Grassmann embedding for Γ_n^h is the same map e_{gr} as given in (3), but it has the property that the image $(\wedge^n V)^\sigma$ can be viewed as a Baer subspace over the fixed field \mathbb{F}^σ . Again, $(\wedge^n V)^\sigma$, as a vector space over \mathbb{F}^σ has dimension $\binom{2n}{n}$, but its vectors do span $\wedge^n V$ (see e.g. [16]). In this paper we will call $e_{\text{gr}} = e_n^\sigma$ and $V_{\text{gr}} = (\wedge^n V)^\sigma$. For the Grassmann embedding of Γ_n^h , we have the following.

Proposition 3.3. (See [16]) *The Grassmann embedding $(e_{\text{gr}}, V_{\text{gr}})$ for Γ_n^h weakly embeds Γ_n^h into $\text{PG}(\wedge^n V)$; the codomain V_{gr} is a Baer subspace of dimension $\binom{2n}{n}$ over \mathbb{F}^σ that spans $\wedge^n V$.*

4. TWO EXTENSION RESULTS

The main results of this section are Propositions 4.9 and 4.10. They allow us to create new generating sets from old. In both, we consider a non-degenerate subspace W of V of codimension 2 and Witt index $n-1$. Proposition 4.9 describes a relation between $\Gamma_2(W)$ and $\Gamma_2(V)$ and Proposition 4.10 describes a relation between $\Gamma_{n-1}(W)$ and $\Gamma_{n-1}(V)$.

Our first goal shall be to establish the existence of sufficiently many “parallel lines” (see Lemmas 4.6 and 4.7). This will be used to prove Corollary 4.8 which says that subspaces of Γ_2 have connected complements. Proposition 4.9 then follows easily from Corollary 4.8. Proposition 4.10 is independent of these results.

Let $\Theta = (\mathcal{P}, \mathcal{L})$ be a partial linear space. Let $d(\cdot, \cdot)$ be the distance relation on \mathcal{P} induced by the natural distance in the collinearity graph of Θ and suppose Θ has diameter $d \in \mathbb{N}$.

Definition 4.1. Let (\mathcal{D}, \prec) be a partially ordered set with unique minimal element 0. A (\mathcal{D}, \prec) -valued distance of Θ is a pair (δ, μ) of maps such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P} \times \mathcal{P} & \xrightarrow{\delta} & \mathcal{D} \\ & \searrow d & \downarrow \mu \\ & & \{0, 1, \dots, d\} \end{array}$$

and we have

- (δ) for any $p, q, r \in \mathcal{P}$, if q and r are collinear, then $\delta(p, q) \prec \delta(p, r)$, $\delta(p, q) = \delta(p, r)$, or $\delta(p, q) \succ \delta(p, r)$,
- (μ) μ weakly preserves order: for any $\alpha, \beta \in \mathcal{D}$, $\alpha \prec \beta$ implies $\mu(\alpha) \leq \mu(\beta)$,

For each $i \in \{0, 1, \dots, d\}$ and $\beta \in \mathcal{D}$, define

$$\begin{aligned} D_i &= \{(p, q) \mid d(p, q) = i\}; \\ \Delta_\beta &= \{(p, q) \mid \delta(p, q) = \beta\} \end{aligned}$$

The following important observation is easily verified.

Lemma 4.2. The partition $\{\Delta_\beta\}_{\beta \in \mathcal{D}}$ of $\mathcal{P} \times \mathcal{P}$ refines $\{D_i\}_{i \in \{0, 1, \dots, d\}}$. More precisely, $\Delta_\beta \subseteq D_{\mu(\beta)}$ for all $\beta \in \mathcal{D}$.

Definition 4.3. We call Θ gated with respect to (δ, μ) if, given any point p and line l , the set $\delta(p, l) = \{\delta(p, q) \mid q \in l\} \subseteq \mathcal{D}$ has a unique minimal element. Moreover, if $\delta(p, l)$ has at least two elements, then the unique minimal element equals $\delta(p, q)$ for a unique point $q \in l$. We then call q the projection of p onto l and write $q = \text{proj}_l(p)$. Note here that in view of (δ) we then have $\delta(p, q) \prec \delta(p, r)$ for all $r \in l - \{q\}$.

We shall henceforth only concern ourselves with distances that are symmetric in the sense that for any $p, q \in \mathcal{P}$ we have $\delta(p, q) = \delta(q, p)$.

Definition 4.4. Let Θ and (\mathcal{D}, \prec) be as above and let (δ, μ) be a symmetric distance. We say that two lines l and m in Θ are parallel if

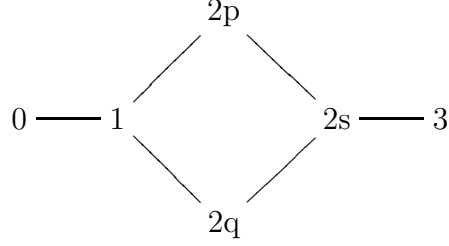


FIGURE 1. The Hasse diagram for (\mathcal{D}, \prec) (rotated 90° clockwise).

- (par-i) $\delta(l, m) = \{\delta(p, q) \mid p \in l, q \in m\}$ has exactly two elements, and
 (par-ii) $\text{proj}_m: l \rightarrow m$ and $\text{proj}_l: m \rightarrow l$ are mutually inverse bijections.

We now turn to the collinearity graph Δ of Γ_2 and let $d(\cdot, \cdot)$ denote the corresponding numerical distance between points. Since any two objects from Γ belong to a common apartment, we can consider a single apartment to determine that two points $x, y \in \Gamma_2$ must be in one of the following two-point relations to one another:

$\delta(x, y)$	description	$d(x, y)$
0	$x = y$	0
1	$\langle x, y \rangle_V$ is a t.i. 3-space	1
2p	$\langle x, y \rangle_V$ is a t.i. 4-space	2
2q	$\langle x, y \rangle_V$ is a non- t.i. 3-space	2
2s	$\langle x, y \rangle_V$ is a 4-space with a radical of dimension 2	2
3	$\langle x, y \rangle_V$ is a non-degenerate 4-space	3

TABLE 1.

Here, 2p stands for “parabolic”, 2q stands for “quadrangular”, and 2s stands for “special”. Note that relation 2p only occurs if $n \geq 4$. We now define a (\mathcal{D}, \prec) -valued distance (δ, μ) on Γ_2 . Let $\mathcal{D} = \{0, 1, 2p, 2q, 2s, 3\}$. The partial order on \mathcal{D} is given by the Hasse diagram in Figure 1. For any two points x, y we let $\delta(x, y)$ be given as in Table 1 and we let μ assign $d(x, y)$ to $\delta(x, y)$ as given there. It is apparent from Table 1 that (δ, μ) satisfies condition (μ) . Condition (δ) is verified by Lemma 4.5. Moreover, it is clear from middle column in Table 1 that (δ, μ) is symmetric.

We now show that the (\mathcal{D}, \prec) -valued distance (δ, μ) is gated in the sense defined above.

Lemma 4.5. *Let l be a line of Γ_2 and p a point not incident with l . Then the number of relations represented by $\{(p, q) \mid q \in l\}$ is at most*

two. Moreover, if two relations occur then there is a unique point q on l such that $(p, q) \prec (p, r)$ for all $r \in l \setminus \{q\}$.

Proof. Let l be a line of Γ_2 and p a point of Γ_2 . We determine the relations between p and the points of l . In doing so we will identify l with a flag (P, Π) where P is an isotropic point, Π is a totally isotropic plane and $P \subset \Pi$ and identify p with a totally isotropic line of the space V . The possible relations between (P, Π) and p are as follows:

- i. $P \subset p \subset \Pi$. In this case, $\delta(p, p) = 0$ and $\delta(p, t) = 1$ for all $t \in l \setminus \{p\}$.
- ii. $P \not\subset p \subset \Pi$. In this case, $\delta(p, t) = 1$ for all $t \in l \setminus \{p\}$.
- iii. $\Pi \cap p = P, \Pi \subset p^\perp$. Then $\delta(p, t) = 1$ for all $t \in l$.
- iv. $\Pi \cap p = P, \Pi \cap p^\perp = r$. Then $\delta(p, r) = 1$ and $\delta(p, t) = 2q$ for all $t \in l \setminus \{r\}$.
- v. $\Pi \cap p = Q \neq P, \Pi \subset p^\perp$. If $r = \langle P, Q \rangle_V$ then $\delta(p, r) = 1$ and $\delta(p, t) = 2p$ for all $t \in l \setminus \{r\}$.
- vi. $\Pi \cap p = Q \neq P, \Pi \cap p^\perp = \langle P, Q \rangle_V = r$. Then $\delta(p, r) = 1$ and $\delta(p, t) = 2s$ for all $t \in l \setminus \{r\}$.
- vii. $\Pi \cap p = Q \neq P, P \not\subset \Pi \cap p^\perp$. If $r = \langle P, Q \rangle_V$ then $\delta(p, r) = 2q$ and $\delta(p, t) = 2s$ for all $t \in l \setminus \{r\}$.
- viii. $\Pi \cap p = 0, \Pi \subset p^\perp$. Then $\delta(p, t) = 2p$ for all $t \in l$.
- ix. $\Pi \cap p = 0, \Pi \cap p^\perp = r \in l$. Then $\delta(p, r) = 2p$ and $\delta(p, t) = 2s$ for all $t \in l \setminus \{r\}$.
- x. $\Pi \cap p = 0, \Pi \cap p^\perp = x$ is a point of Γ_2 not on l . Now $\delta(p, t) = 2p$ for all $t \in l$.
- xi. $\Pi \cap p = 0, \Pi \cap p^\perp = P$. Now $\delta(p, t) = 2s$ for all $t \in l$.
- xii. $\Pi \cap p = 0, \Pi \cap p^\perp = Q \neq P$. If $r = \langle P, Q \rangle_V$ then $\delta(p, r) = 2s$ and $\delta(p, t) = 3$ for all $t \in l \setminus \{r\}$. \square

Since by Lemma 4.5, Γ_2^f is gated in the sense of Definition 4.3, it makes sense to talk about parallel lines.

Lemma 4.6. *Given any two non-collinear points p and q , there exist parallel lines l and m on p and q so that $\text{proj}_m(p) \neq q$.*

Proof. Let $\mathcal{E} = \{e_i, f_i \mid i = 1, 2, \dots, n\}$ be a hyperbolic basis such that $p, q \in \Sigma(\mathcal{E})$ with $p = \langle e_1, e_2 \rangle$. Now $l = (p \cap p', \langle p, p' \rangle)$ and $m =$

$(q \cap q', \langle q, q' \rangle)$ are given by the following table, where $q' = \text{proj}_m(p)$ and $p' = \text{proj}_l(q)$.

q	p'	q'	$\delta(p, q) = \delta(p', q')$	$\delta(p, q') = \delta(p', q)$
$\langle f_1, f_2 \rangle$	$\langle e_1, e_3 \rangle$	$\langle f_1, f_3 \rangle$	3	2s
$\langle f_1, f_3 \rangle$	$\langle e_1, f_3 \rangle$	$\langle f_1, e_2 \rangle$	2s	2q
$\langle f_1, e_2 \rangle$	$\langle e_2, f_3 \rangle$	$\langle e_2, e_3 \rangle$	2q	1
$\langle e_3, e_4 \rangle$	$\langle e_1, e_3 \rangle$	$\langle e_2, e_4 \rangle$	2p	1

□

Note that in Lemma 4.6 we do not require that, given any two points p and q and a line l on p , there exists a line m on q that is parallel to l . This is not true even in this geometry. However, we do not need that for proving Lemma 4.7.

The first author learned the following useful argument from Andries Brouwer.

Lemma 4.7. *Let Θ be a thick partial linear space with point-set \mathcal{P} of finite diameter whose two-point relations can be ordered in some way so that given any two points p and q , there exist parallel lines l and m on p and q so that $\text{proj}_m(p) \neq q$. Then the complement of any proper subspace is connected.*

Proof. Let \mathcal{H} be a proper subspace and set $\mathcal{A} = \mathcal{P} - \mathcal{H}$.

For any two points $p, q \in \mathcal{A}$ we show that there is a path of points and lines in \mathcal{A} from p to q .

If p and q are on a common line l , then p, l, q is a path in \mathcal{A} connecting p and q and we are done.

Now let p and q be non-collinear, and let l and m be the parallel lines assumed to exist. Then because lines of Θ are thick and intersect the subspace \mathcal{H} in none, one, or all of their points, one of the following must happen: (1) $r = \text{proj}_m(p) \in \mathcal{A}$, (2) $s = \text{proj}_l(q) \in \mathcal{A}$, or (3) there exist points $t, u \in \mathcal{A}$ different from p, q, r, s and on l and m , respectively, such that $\text{proj}_m(t) = u$.

In all cases, p and q are collinear to points of \mathcal{A} that are nearer to each other than are p and q (since $\text{proj}_m(p) \neq q$). Since the diameter of Θ is finite, by repeating this argument we find a path in \mathcal{A} connecting p to q . □

From Lemmas 4.6 and 4.7, we obtain the following result.

Corollary 4.8. *The complement of a proper subspace of Γ_2 is connected.*

Proposition 4.9. *Assume $n \geq 3$ and let W be a non-degenerate $2(n-1)$ -space of Witt index $n-1$ in V . Then $\mathcal{H} = \{K \in \mathcal{P}(\Gamma_2^f) \mid K \cap W \neq \{0\}\}$ is a hyperplane of Γ_2^f . This is a maximal subspace.*

Proof. Let the line l of Γ_2^f consist of all 2-objects K with $A \subseteq K \subseteq C$ for some 1-object A and 3-object C . Now $\langle A, A^\perp \cap W \rangle / A$ is a hyperplane or all of the polar space A^\perp / A . It follows that, accordingly, either one or all of the points of l meet W , which proves the lemma. The latter claim follows from Corollary 4.8. \square

Proposition 4.10 can be used for an inductive argument on the Witt index n of V .

Proposition 4.10. *Assume $\dim V \geq 6$. Let $W \leq V$ be a non-degenerate subspace of codimension 2 and Witt index $n-1$, and let P, Q be two isotropic points in the hyperbolic line W^\perp . If S is a set of points in $\Gamma_k(W)$, then*

$$\langle S \rangle_{\Gamma_k(W)} \subseteq \langle S, \text{Res}_k(P), \text{Res}_k(Q) \rangle_{\Gamma_k(V)}.$$

Proof. Clearly points of $\Gamma_k(W)$ are also points of $\Gamma_k(V)$. If $k < n-1$, then also lines of the former are lines of the latter. In that case the result is trivial.

Now let $k = n-1$. Note that W has Witt index $(n-1)$ and V has Witt index n . Let $x, y \in \Gamma_{n-1}(W)$ be on some line l of $\Gamma_{n-1}(W)$. Then there is a t.i. $(n-2)$ -space $L \leq W$ such that l consists of all t.i. $(n-1)$ -spaces in $L^\perp \cap W$. Note that $L^\perp \cap W$ is a non t.i. n -space and so the points on l are not collinear in $\Gamma_{n-1}(V)$.

However, all points of l are contained in the residue $\text{Res}_{\Gamma_{n-1}}(L) \cong \Gamma_1(L^\perp/L)$. Thus, by Proposition 2.1 this residue is generated as a subspace of $\Gamma_{n-1}(V)$ by the set $\{x, y, \langle L, P \rangle_V, \langle L, Q \rangle_V\}$. We are done since $\langle L, P \rangle \in \text{Res}_{n-1}(P)$ and $\langle L, Q \rangle \in \text{Res}_{n-1}(Q)$. \square

5. GENERATING SETS

We inductively define the following set $\mathcal{S}_{n,k}^f = \mathcal{S}_{n,k}^f(V)$ of k -objects in Γ_k^f : Consider a non-degenerate $2(n-1)$ -space W of Witt index $n-1$ of V and let P and Q form a hyperbolic pair in W^\perp . We formally define $\mathcal{S}_{n,0}(V) = \{0\}$, thus containing one element. Let $\mathcal{S}_{n,1}(V)$ and $\mathcal{S}_{n,n}(V)$ be the generating sets of Γ_1^f and Γ_n^f respectively. The existence of these sets is guaranteed by Proposition 2.1 and Theorem 2.3. Now assume $2 \leq k \leq n-1$. Using induction on n , we choose $\mathcal{S}_{n-1,l}^f(W)$ for $l = k, k-1, k-2$. Let

$$\widehat{\mathcal{S}}_{n-1,k-1}^f(W) = \{\langle L, P \rangle, \langle L, Q \rangle \mid L \in \mathcal{S}_{n-1,k-1}^f(W)\}.$$

Finally we describe the set $\widehat{\mathcal{S}}_{n-1,k-2}^f(W)$. Let M be any $(k-2)$ -object in W . Then, the collection of all k -objects K containing M and intersecting $M^\perp \cap W$ non-trivially, is a hyperplane in the residue of M in the k -Grassmannian (compare Proposition 4.9). Let \widehat{M} be some k -object on M , not contained in this hyperplane. Now set

$$\widehat{\mathcal{S}}_{n-1,k-2}^f(W) = \{\widehat{M} \mid M \in \mathcal{S}_{n-1,k-2}^f(W)\}.$$

Using induction on n we now define

$$\mathcal{S}_{n,k}^f(V) = \mathcal{S}_{n-1,k}^f(W) \uplus \widehat{\mathcal{S}}_{n-1,k-1}^f(W) \uplus \widehat{\mathcal{S}}_{n-1,k-2}^f(W).$$

We note that in case $k = 2$, $\widehat{\mathcal{S}}_{n-1,k-2}^f$ contains one element.

Lemma 5.1.

- (a) For $1 \leq k \leq m$ integers, we have $\binom{m}{k} = \binom{m-2}{k} + 2\binom{m-2}{k-1} + \binom{m-2}{k-2}$.
- (b) For $1 \leq k \leq n$ integers, we have

$$\begin{aligned} \binom{2n}{k} &= \binom{2(n-1)}{k} + 2\binom{2(n-1)}{k-1} + \binom{2(n-1)}{k-2}, \text{ and} \\ \binom{2n}{k} - \binom{2n}{k-2} &= \binom{2(n-1)}{k} + 2\binom{2(n-1)}{k-1} - 2\binom{2(n-1)}{k-3} - \binom{2(n-1)}{k-4}. \end{aligned}$$

Proof. (a) Apply the binomial theorem to $(x+1)^m = (x+1)^2 \cdot (x+1)^{m-2}$ and compute the coefficient of x^k . (b) This is immediate from (a). \square

Corollary 5.2. Let $1 \leq k \leq n$ be integers. Then,

- (a) $|\mathcal{S}_{n,k}^s| = \binom{2n}{k} - \binom{2n}{k-2}$,
- (b) $|\mathcal{S}_{n,k}^h| = \binom{2n}{k}$.

Proof. For $k = 1, n$, this is clear by Proposition 2.1 and Theorem 2.3. Now let $k \geq 2$. By construction, we have $|\mathcal{S}_{n,k}^f| = |\mathcal{S}_{n-1,k}^f| + 2|\mathcal{S}_{n-1,k-1}^f| + |\mathcal{S}_{n-1,k-2}^f|$. Therefore the lemma follows by induction on $2 \leq k \leq n-1$ and for each k by induction on $n > k$ from Lemma 5.1. \square

We recall the following definition from [7]. Given a set \mathcal{S} of points in Γ_k^f , we call an object X of Γ \mathcal{S} -full if $\text{Res}_k(X) \leq \langle \mathcal{S} \rangle_{\Gamma_k^f}$.

Proposition 5.3. Let $1 \leq k \leq n$ be integers. Then,

$$\langle \mathcal{S}_{n,k}^f \rangle_{\Gamma_k^f} = \Gamma_k^f.$$

Proof. The case $k = 1$ follows from Proposition 2.1. The case $k = n$ follows from Theorem 2.3. We now continue by induction on n . Set $\mathcal{S} = \mathcal{S}_{n,k}^f(V)$. Recall W is a non-degenerate $2(n-1)$ -dimensional subspace of V of Witt index $n-1$ and let P and Q be two points of the hyperbolic line W^\perp . Let $\mathcal{S}_l(W) = \mathcal{S}_{n-1,l}^f(W)$.

We first note that the points P and Q are \mathcal{S} -full. Let us see why this is so. For each $L \in \mathcal{S}_{n-1,k-1}^f(W)$, we have $\langle L, P \rangle \in \mathcal{S}$. Now let L_1, L_2

be $(k-1)$ -objects that are on some line of $\Gamma_{k-1}^f(W)$. Note that, since $k-1 < n-1$, this line is of the form (A, B) for some $(k-2)$ -object A and k -object B with $A \leq B \leq W$. Then, $(\langle A, P \rangle, \langle B, P \rangle)$ is a line of $\Gamma_k^f(V)$ containing the points $\langle L_1, P \rangle$ and $\langle L_2, P \rangle$. It follows that for each $L \in \Gamma_{k-1}^f(W) = \langle \mathcal{S}_{k-1}(W) \rangle$ we have $\langle L, P \rangle \in \langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$. In particular, P and, similarly, Q are \mathcal{S} -full. It now follows from Proposition 4.10 that

$$\langle \mathcal{S}_k(W) \rangle_{\Gamma_k^f(W)} \subseteq \langle \mathcal{S}_k(W) \rangle_{\Gamma_k^f(V)}.$$

We now show that every $(k-1)$ -object $L \in \Gamma_{k-1}^f(W)$ is \mathcal{S} -full. The residue of L is of type $\Gamma_1^f(L^\perp/L)$. Now by the preceding, $\langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$ contains all points of $\Gamma_1^f(L^\perp \cap W/L)$ as well as the points $\langle P, L \rangle$ and $\langle Q, L \rangle$. Therefore, by the case $k=1$ (Proposition 2.1) $\langle \mathcal{S} \rangle$ contains all points of $\Gamma_1^f(L^\perp/L)$.

We'll now show that every $(k-2)$ -object $M \in \Gamma_{k-2}^f(W)$ is \mathcal{S} -full. The residue of M is of type $\Gamma_2^f(M^\perp/M)$. Now by the preceding, $\langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$ contains all 2-objects of $\Gamma_2^f(M^\perp/M)$ meeting $(M^\perp \cap W)/M$ non-trivially. Therefore, by Proposition 4.9, it suffices to show that $\langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$ also contains one k -object K with $M = K \cap W$. For each $M \in \mathcal{S}_{k-2}(W)$, this is true by definition of \mathcal{S} . Note that this settles the $k=2$ case.

From now on we may assume $k \geq 3$. Suppose that $M \in \Gamma_k^f(W) \setminus \mathcal{S}_{k-2}(W)$. Now let M_1, M_2 be \mathcal{S} -full $(k-2)$ -objects that are on some line of $\Gamma_{k-2}^f(W)$ containing M . Note that, since $k-2 < n-1$, this line is of the form (A, B) for some $(k-3)$ -object A and $(k-1)$ -object B with $A \leq B \leq W$. Since $k \leq n-1$, there exists a $(k+1)$ -object of the form $\langle B, R \rangle$ where R is of dimension 2 and disjoint from W . Now $(\langle A, R \rangle, \langle B, R \rangle)$ is a line of $\Gamma_k^f(V)$ spanned by the points $\langle M_1, R \rangle$ and $\langle M_2, R \rangle$. Since these points belong to $\langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$, so does $\langle M, R \rangle$. Since $M = \langle M, R \rangle \cap W$, we find that M is \mathcal{S} -full.

Since, for each $l = k-2, k-1, k$, every l -object in W is \mathcal{S} -full, and every k -object intersects W in an object of type $k-2, k-1$, or k , we find that $\Gamma_k^f(V) = \langle \mathcal{S} \rangle_{\Gamma_k^f(V)}$. \square

Proposition 5.4. *The image of $\mathcal{S}_{n,k}^f(V)$ under e_{gr} forms a basis for V_{gr} .*

Proof. Let $I = \{1, 2, \dots, 2n\}$. Given any basis $\{e_i \mid i \in I\}$ for V , it is known that $\bigwedge^k V$ has basis $\{e_K \mid K \subseteq I, |K| = k\}$, where $e_K = \bigwedge_{k \in K} e_k$ is taken in order of increasing k . Let W, P and Q be as in the construction of $\mathcal{S}_{n,k}^f(V)$. Choosing our basis so that $\{e_i \mid i \in I - \{2n-1, 2n\}\}$ is a basis for W and $P = \langle e_{2n-1} \rangle$ and $Q = \langle e_{2n} \rangle$ it

follows that

$$(6) \quad \bigwedge^k V = \bigwedge^k W \oplus \left(\bigwedge^{k-1} W \wedge P \right) \oplus \left(\bigwedge^{k-1} W \wedge Q \right) \oplus \left(\bigwedge^{k-2} W \wedge P \wedge Q \right).$$

We wish to show that for all $n \geq 2$ and $1 \leq k \leq n$, the images of $\mathcal{S}_{n,k}^f$ under the embedding e_{gr} are linearly independent. We note that in these cases the geometry Γ_k^f is embedded into a subspace of $\bigwedge^k V$. Therefore it suffices to show that these images are linearly independent in $\bigwedge^k V$. With slight abuse of language, we shall say that a set \mathcal{S} of points of Γ_k^f is “ e_{gr} -independent” if the set $e_{\text{gr}}(\mathcal{S})$ is linearly independent in $\bigwedge^k V$.

We show that $\mathcal{S}_{n,k}^f(V)$ is e_{gr} -independent, using induction on n and, for each n we distinguish cases $k = 1$, $k = n$ and $2 \leq k \leq n - 1$. If $k = 1$ and $n \geq 1$, then $V_{\text{gr}} = V$ and $\mathcal{S}_{n,1}^f$ is simply the set of 1-spaces spanned by a hyperbolic basis for V , so the claim holds.

Next, we address the special case $n = k$. If $f = s$, we note that Γ_n^s is naturally embedded into $\bigwedge^n V$ and so the claim follows directly from Theorem 2.3. From Proposition 3.3 we know that the geometry Γ_n^h is weakly, but not fully embedded into $\bigwedge^n V$ since this geometry and its embedding are naturally defined over the fixed field \mathbb{F}^σ of \mathbb{F} under σ . However, we can still view the image of $\mathcal{S}_{n,n}^h$ under e_{gr} as a set of vectors of $\bigwedge^n V$. We then note that $V_{\text{gr}} \otimes_{\mathbb{F}^\sigma} \mathbb{F} = \bigwedge^n V$, so in particular, by Theorem 2.3, the image $\mathcal{S}_{n,n}^h$ under e_{gr} spans $\bigwedge^n V$.

If we view $\mathcal{S}_{n-1,n-1}^h(W)$ as a subset of $\mathcal{S}_{n,n-1}(V)$, then its image under e_{gr} is linearly independent in V_{gr} by Theorem 2.3. By the preceding paragraph, $e_{\text{gr}}(\mathcal{S}_{n-1,n-1}^h(W)) = \{\langle e_{\text{gr}}(s) \rangle_{\mathbb{F}} \mid s \in \mathcal{S}_{n-1,n-1}^h(W)\}$. And since the codomain W_{gr} of the embedding e_{gr} of $\Gamma_{n-1}(W)$ linearly spans $\bigwedge^{n-1} W$, the spanning set $\mathcal{S}_{n-1,n-1}^h(W)$ is e_{gr} independent. This settles the base case $k = n$.

Next, assume $2 \leq k \leq n - 1$. We note that

$$\mathcal{S}_{n,k}^f(V) = \mathcal{S}_{n-1,k}^f(W) \uplus \widehat{\mathcal{S}}_{n-1,k-1}^f(W) \uplus \widehat{\mathcal{S}}_{n-1,k-2}^f(W).$$

Now assume that $\mathcal{S}_{n-1,k}^f(W)$, $\mathcal{S}_{n-1,k-1}^f(W)$ and $\mathcal{S}_{n-1,k-2}^f(W)$ are all independent. Note that

$$\begin{aligned} e_{\text{gr}}(\mathcal{S}_{n-1,k}^f(W)) &\subseteq \bigwedge^k W, \\ e_{\text{gr}}(\widehat{\mathcal{S}}_{n-1,k-1}^f(W)) &\subseteq \left(\bigwedge^{k-1} W \wedge P \right) \oplus \left(\bigwedge^{k-1} W \wedge Q \right), \\ e_{\text{gr}}(\widehat{\mathcal{S}}_{n-1,k-2}^f(W)) &\subseteq \bigwedge^{k-2} W \wedge P \wedge Q. \end{aligned}$$

Since $\mathcal{S}_{n-1,k-1}^f(W)$ is independent, so is the image of $\widehat{\mathcal{S}}_{n-1,k-1}^f(W)$ in $\bigwedge^{k-1} W \wedge P$, and likewise for Q . For the same reason, the image of $\widehat{\mathcal{S}}_{n-1,k-2}^f(W)$ in $\bigwedge^{k-2} W \wedge P \wedge Q$ is independent. The direct sum decomposition (6) shows that the set $\widehat{\mathcal{S}}_{n-1,k-1}^f(W)$ is independent, and that the sets $\mathcal{S}_{n-1,k}^f(W)$, $\widehat{\mathcal{S}}_{n-1,k-1}^f(W)$, and $\widehat{\mathcal{S}}_{n-1,k-2}^f(W)$ are also pairwise independent. \square

Proposition 5.5. *Let $1 \leq k \leq n$. The Grassmann embedding $(e_{\text{gr}}, V_{\text{gr}})$ for Γ_k^s has dimension $\binom{2n}{k} - \binom{2n}{k-2}$.*

Proof. In Proposition 5.4 we showed directly that the image of the generating set $\mathcal{S}_{n,k}^s(V)$ forms a basis for V_{gr} . It then follows from Corollary 5.2 that $\dim(V_{\text{gr}}) = |\mathcal{S}_{n,k}^s(V)| = \binom{2n}{k} - \binom{2n}{k-2}$. \square

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